§26 Compact Spaces

Definition Let X be a set topological space and let $\mathscr{F} = \{U \mid U \subseteq X\}$ be a collection of subsets of X. Then $\mathscr{F}$ is called a *cover* of X if $\bigcup_{U \in \mathscr{F}} U = X$.

• $\mathscr{F}$ is called an *open cover* of X if every $U \in \mathscr{F}$ is an open subset of X, and $\bigcup_{U \in \mathscr{F}} U = X$.

• $\mathscr{F}'$ is called a *subcover* of $\mathscr{F}$ if $\mathscr{F}' \subseteq \mathscr{F}$, and $\bigcup_{U \in \mathscr{F}'} U = X$.

• $\mathscr{F}'$ is called a *finite subcover* of $\mathscr{F}$ if $\mathscr{F}'$ is a subcover of $\mathscr{F}$, and $\mathscr{F}'$ contains finite number of elements of $\mathscr{F}$.

Definition A space X is said to be *compact* if every open covering $\mathscr{F} = \{U \mid U \subseteq X\}$ of X has a *finite subcover* $\mathscr{F}' = \{U_i \in \mathscr{F} \mid 1 \leq i \leq n\}$.

Example 1. The real line $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ by open intervals

$$\mathscr{F} = \{(n,\, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection that covers $\mathbb{R}$.

Example 2. The following subspace of $\mathbb{R}$ is compact:

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$

Given an open covering $\mathscr{F}$ of X, there is an element U of $\mathscr{F}$ containing 0. The set U contains all but finitely many of the points $1/n$; choose, for each point of X not in U, an element of $\mathscr{F}$ containing it. The collection consisting of these elements of $\mathscr{F}$, along with the element U, is a finite subcollection of $\mathscr{F}$ that covers X.

Example 3. Any space X containing only finitely many points is necessarily compact, because in this case every open covering of X is finite.

Example 4. The interval $(0,\, 1]$ is not compact; the open covering

$$\mathscr{F} = \{(1/n,\, 1] \mid n \in \mathbb{Z}_+\}$$

contains no finite subcollection covering $(0,\, 1]$. Nor is the interval $(0,\, 1)$ compact; the same argument applies. On the other hand, the interval $[0,\, 1]$ is compact; you are probably already familiar with this fact from analysis. In any case, we shall prove it shortly.

In general, it takes some effort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones. Then in the next section we shall show certain specific spaces are compact. These spaces include all closed intervals in the real line, and all closed and bounded subsets of $\mathbb{R}^n$.

Let us first prove some facts about subspaces. If Y is a subspace of X, a collection $\mathscr{F}$ of subsets of X is said to *cover Y* if the union of its elements contains Y.

Lemma 26.1 Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof Suppose that Y is compact and $\mathscr{F} = \{U_\alpha \mid \alpha \in J\}$ is a covering of Y by sets open in X. Then the collection

$$\{U_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{U_{\alpha_1} \cap Y, \dots, U_{\alpha_n} \cap Y\}$$

covers Y. Then $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a subcollection of $\mathscr{F}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathscr{A} = \{A_\alpha \mid \alpha \in J\}$ be a covering of Y by sets open in Y. For each α, choose a set U_α open in X such that

$$A_\alpha = U_\alpha \cap Y, \quad \text{where } U_\alpha \text{ is open in } X \text{ for each } \alpha \in J$$

The collection $\mathscr{F} = \{U_\alpha \mid \alpha \in J\}$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers Y. Then $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ is a subcollection of $\mathscr{A}$ that covers Y.

Theorem 26.2 Every closed subspace of a compact space is compact, that is, if C is a closed subset of a compact topological space X, then C is compact.

Proof Let $\mathscr{F} = \{U_\alpha \mid \alpha \in J\}$ be a family of open subsets of X that covers C, i.e.

$$C \subseteq \bigcup_{\alpha \in J} U_\alpha.$$

Since $(X \setminus C) \cup \mathscr{F}$ is an open cover of X and since X is compact, there exist $U_1, \dots, U_n \in \mathscr{F}$ such that

$$C \cup (X \setminus C) = X = \left(\bigcup_{i=1}^{n} U_i \right) \cup (X \setminus C) \implies C \subseteq \bigcup_{i=1}^{n} U_i$$

and $\{U_i \mid 1 \le i \le n\}$ is a finite subcover of $\mathscr{F}$. This shows that C is compact.

Theorem 26.3 Every compact subspace of a Hausdorff space is closed, that is, if Y is a compact subspace of a Hausdorff space X, then Y is closed.

Proof Let x_0 be a point of $X \setminus Y$. For each point y of Y, since X is Hausdorff, there exist disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively. Since $\{V_y \mid y \in Y\}$ is an open covering Y and Y is compact, there are finitely many of them $V_{y_1}, \dots, V_{y_n}$ cover Y. Let

$$V = V_{y_1} \cup \cdots \cup V_{y_n} \quad \text{and} \quad U = U_{y_1} \cap \cdots \cap U_{y_n} \implies Y \subset V,\ x_0 \in U \text{ and } U \cap V = \emptyset$$

since if $z \in V \implies z \in V_{y_i}$ for some i, hence $z \notin U_{y_i} \implies z \notin U$ (see Figure 26.1). This implies that U is an open neighborhood of x_0 in $X \setminus Y$, so $X \setminus Y$ is open in X and Y is closed.

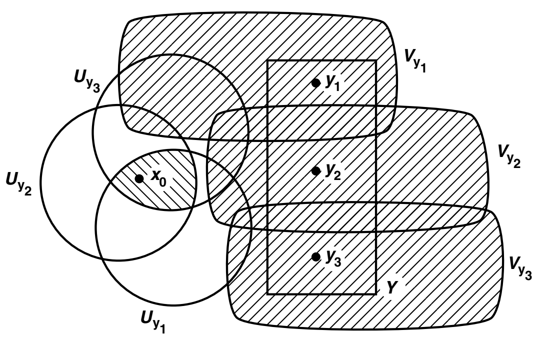


Figure 26.1

Lemma 26.4 If C is a compact subset of a Hausdorff space X and $x \in X \setminus C$, then there exist disjoint neighborhoods U and V of x and C, respectively. Therefore a compact subset of a Hausdorff space is closed.

Proof For each $z \in C$, since X is Hausdorff, let U_z and V_z be disjoint open subsets such that $x \in U_z$ and $z \in V_z$. Since

$$C \subseteq \bigcup_{z \in C} V_z,$$

$\mathscr{F} = \{V_z \mid z \in C\}$ is an open cover of C, and since C is compact there exist a finite subcover $\{V_{z_i} \mid z_i \in C, \text{ for each } 1 \leq i \leq n\}$ of $\mathscr{F}$ such that

$$C \subseteq \bigcup_{i=1}^{n} V_{z_i}.$$

Let $V = \bigcup_{i=1}^{n} V_{z_i}$. Since $V_{z_i} \cap U_{z_i} = \emptyset$ and $x \in U_{z_i}$ for each $1 \leq i \leq n$, the sets $U = \bigcap_{i=1}^{n} U_{z_i}$ and V are disjoint open neighborhoods of x and C.

Theorem 26.5 If X is a compact topological space and $f : X \to Y$ is a continuous function, then $f(X)$ is a compact subspace of Y.

Proof Let $\mathscr{F} = \{V_\alpha \mid \alpha \in J\}$ be a covering of the set $f(X)$ by sets open in Y. Since $f : X \to Y$ is continuous, the collection

$$\{f^{-1}(V_\alpha) \mid \alpha \in J\}$$

is an open covering X. Hence finitely many of them, say

$$f^{-1}(V_1), \ldots, f^{-1}(V_n),$$

cover X. Then the sets $V_1, \ldots, V_n$ cover $f(X)$.

Remark Theorem 26.5 implies that the compactness is a topological property, i.e. if X is compact and if Y is homeomorphic to X, then Y is compact.

Theorem 26.6 If X is a compact space, Y is a Hausdorff space and $f : X \to Y$ is a bijective continuous function, then $f : X \to Y$ is a homeomorphism.

Proof If C is a closed subset of X, since X is compact and $f : X \to Y$ is continuous, then C is compact in X and $f(C)$ is compact in Y. Since Y is Hausdorff and $f : X \to Y$ has a one-to-one, onto, the set $\left(f^{-1}\right)^{-1}(C) = f(C)$ is closed in Y for each closed subset C of X which implies that $f^{-1} : Y \to X$ is continuous and $f : X \to Y$ is a homeomorphism.

Theorem 26.7 The product space $X \times Y$ is compact if and only if both X and Y are compact.

Proof ($\Longrightarrow$) If $X \times Y$ is compact, then both X and Y are compact since the projections $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ are onto and continuous functions.

($\Longleftarrow$) Suppose both X and Y are compact spaces and let $\mathscr{F}$ be an open cover of $X \times Y$ by basic open sets of the form $U \times V$, where U is open in X and V is open in Y.

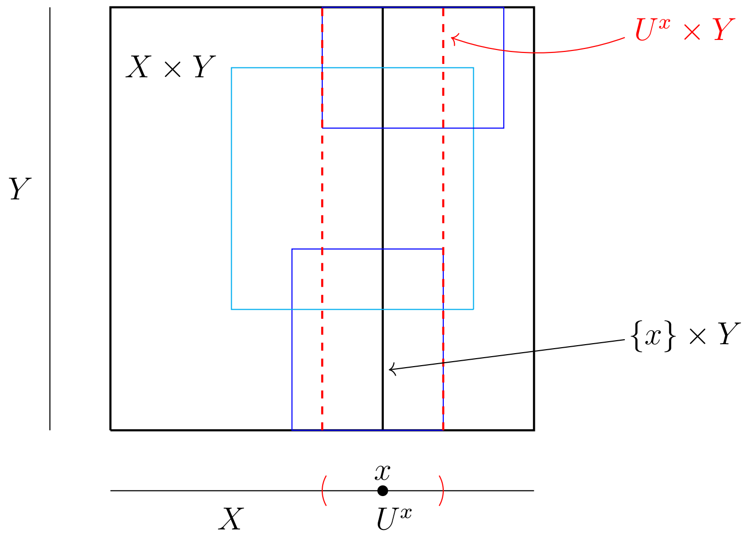
For each $x \in X$, consider the subset $\{x\} \times Y$ of $X \times Y$ with the induced topology. Since the map

$$\pi_2|_{\{x\} \times Y} : \{x\} \times Y \to Y$$

is a homeomorphism and Y is compact, $\{x\} \times Y$ is compact and there exists a finite subfamily $\{U_i^x \times V_i^x \mid 1 \leq i \leq n_x\}$ of $\mathscr{F}$ whose union contains $\{x\} \times Y$. Since $x \in U_i^x$ for each $1 \leq i \leq n_x$,

$$U^x = \bigcap_{i=1}^{n_x} U_i^x \neq \emptyset \quad \text{and} \quad U^x \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x,$$

the union of these sets contains more than $\{x\}$, it actually contains all of $U^x \times Y$.



Since the family $\{U^x \mid x \in X\}$ is an open cover of X, we can select a finite subcover $\{U^{x_j} \mid 1 \le j \le s\}$ of X such that $X = \bigcup_{i=j}^{s} U^{x_j}$ and

$$X \times Y = \bigcup_{j=1}^{s} (U^{x_j} \times Y) \subseteq \bigcup_{j=1}^{s} \bigcup_{i=1}^{n_{x_j}} (U_i^{x_j} \times V_i^{x_j})$$

this implies that $X \times Y$ is compact since it can be covered by a finite subfamily $\{U_i^{x_j} \times V_i^{x_j} \mid 1 \le j \le s,\ 1 \le i \le n_{x_j}\}$ of $\mathscr{F}$.

Definition A collection $\mathscr{C}$ of subsets of X is said to have the *finite intersection property* if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathscr{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9 Let X be a topological space. Then X is compact if and only if for every collection $\mathscr{C} = \{C_\alpha \mid \alpha \in J\}$ of closed sets in X having the finite intersection property satisfies

$$\bigcap_{\alpha \in J} C_\alpha \neq \emptyset.$$

Proof Let $\mathscr{C} = \{C_\alpha \mid \alpha \in J\}$ be a collection of closed subsets in X and let $\mathscr{F} = \{U_\alpha = X \setminus C_\alpha \mid \alpha \in J\}$ be the collection of open subsets in X. Since

- $\emptyset = \bigcap_{\alpha \in J} C_\alpha = \bigcap_{\alpha \in J} (X \setminus U_\alpha) = X \setminus \left(\bigcup_{\alpha \in J} U_\alpha \right) \iff \bigcup_{\alpha \in J} U_\alpha = X \iff \mathscr{F}$ is an open cover of X,
- $\mathscr{C}$ has the finite intersection property $\iff \mathscr{F}$ does not have a finite subcover,

X is not compact if and only if there is a collection $\mathscr{C} = \{C_\alpha \mid \alpha \in J\}$ of closed sets in X satisfying $\bigcap_{\alpha \in J} C_\alpha = \emptyset$ and the finite intersection property.

A special case of this theorem occurs when we have a *nested sequence* $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\{C_n \mid n \in \mathbb{Z}_+\}$ automatically has the finite intersection property. Then the intersection

$$\bigcap_{n \in \mathbb{Z}_+} C_n \neq \emptyset.$$

§27 Compact Subspaces of the Real Line

Definition A set $C \subset \mathbb{R}^n$ is called a bounded subset of $\mathbb{R}^n$ if there exists a ball $B_r(p) = \{x \in \mathbb{R}^n \mid |x - p| < r\}$ or a rectangular box $\prod_{k=1}^{n}[a_k,\, b_k] = [a_1,\, b_1] \times \cdots \times [a_n,\, b_n] \subset \mathbb{R}^n$ such that either $C \subset B_r(p)$ or $C \subset \prod_{k=1}^{n}[a_k,\, b_k]$.

Theorem 27.3 (Heine-Borel Theorem) A subset X of $\mathbb{R}^n$ is closed and bounded if and only if X is compact, that is, every open cover $\mathscr{F}$ of X (with the induced topology) has a finite subcover.

Lemma A closed interval $[a,\, b]$ of the real line $\mathbb{R}$ is compact.

Proof of the Lemma Suppose that the Lemma is false. Let $\mathscr{F}$ be an open cover of $[a,\, b]$ which does not contain a finite subcover.

- Set $I_1 = [a,\, b]$.
- Subdivide $[a,\, b]$ into 2 closed subintervals of equal length $[a,\, (a+b)/2]$ and $[(a+b)/2,\, b]$. At least one of these must have the property that it is not contained in the union of any finite subfamily of $\mathscr{F}$. Select one of $[a,\, (a+b)/2]$, $[(a+b)/2,\, b]$ which has this property and call it I_2.
- Now repeat the process, bisecting I_2 and selecting one half, called I_3, which is not contained in the union of any finite subfamily of $\mathscr{F}$.
- Continuing in this way, we obtain a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \quad \text{with the length of } I_n \text{ equals } |I_n| = \frac{b-a}{2^{n-1}} \quad \forall\, n = 1, 2, \ldots$$

- For each $n \in \mathbb{N}$, let a_n be the left endpoint of I_n, since $\{a_n\}$ is a nondecreasing, bounded above sequence of real numbers, the least upper bound property of $\mathbb{R}$ and $\lim_{n\to\infty} |I_n| = 0$, the sequence $\{a_n\}$ converges and there exists a unique point

$$p = \sup\{a_n \mid n \in \mathbb{N}\} = \lim_{n\to\infty} a_n \in [a,\, b] \quad \text{such that} \quad \bigcap_{n=1}^{\infty} I_n = \{p\}.$$

- Since $p \in [a,\, b]$, there is an open set $O \in \mathscr{F}$, an $\varepsilon > 0$ and an $n \in \mathbb{N}$ such that $p \in O$, $(p-\varepsilon,\, p+\varepsilon) \cap [a,\, b] \subseteq O$ and $|I_n| < \varepsilon$. Also since $a_n \leq p \in I_n$, $I_n \subseteq (p-\varepsilon,\, p+\varepsilon) \cap [a,\, b] \subseteq O$, i.e. I_n is contained in a single element of $\mathscr{F}$, which is a contradiction to the choice of I_n.

Lemma A closed rectangular box $\prod_{k=1}^{n}[a_k,\, b_k] = [a_1,\, b_1] \times \cdots \times [a_n,\, b_n]$ of $\mathbb{R}^n$ is compact.

Theorem 28.1 (Bolzano-Weierstrass Property) An infinite set of points in a compact space must have a limit point, that is, if S is an infinite subset of a compact space X, then $S' \cap X \neq \emptyset$.

Proof Let S be a subset of a compact space X that does not contain any limit point of S, i.e.

$$S' \cap X = \emptyset.$$

For each $x \in X$, since $x \notin S'$, there is an open neighborhood $O(x)$ of x such that

$$O(x) \cap S \setminus \{x\} = \emptyset \implies O(x) \cap S = \begin{cases} \emptyset & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

By the compactness of X, the open cover $\{O(x) \mid x \in X\}$ has a finite subcover. But each set $O(x)$ contains at most one point of S and therefore S must be a finite set.

Theorem 27.4 (Extreme Value Theorem) A continuous real-valued function defined on a compact space is bounded and attains its bounds.

Proof If $f : X \to \mathbb{R}$ is continuous and if X is compact, then $f(X)$ is compact. Therefore $f(X)$ is bounded closed subset of $\mathbb{R}$ by a preceding theorem and there exist $x_1, x_2 \in X$ such that

$$f(x_1) = \sup(f(X)) \quad \text{and} \quad f(x_2) = \inf(f(X)).$$

Definition Let A, B be subsets of the metric space (X, d). Then the diameter of A is defined by

$$\operatorname{diam}(A) = \sup_{x,\, y \in A} d(x, y)$$

and the distance $d(A, B)$ between A and B is defined by

$$d(A, B) = \inf_{x \in A,\, y \in B} d(x, y).$$

Lemma 27.5 (The Lebesgue Number Lemma) Let X be a compact metric space and let $\mathscr{F}$ be an open cover of X. Then there exists a real number $\delta > 0$ (called a Lebesgue number of $\mathscr{F}$) such that any subset of X of diameter less than δ is contained in some member of $\mathscr{F}$.

Proof of the Lebesgue Number Lemma If Lebesgue's Lemma is false, there exists a sequence $\{A_n \neq \emptyset \mid n \in \mathbb{N}\}$ of subsets of X such that

- $A_n \not\subseteq U$ for each $U \in \mathscr{F}$, for each $n \in \mathbb{N}$.
- $d(A_n) = \operatorname{diam}(A_n) \searrow 0$ (diameter of A_n deceases to 0).

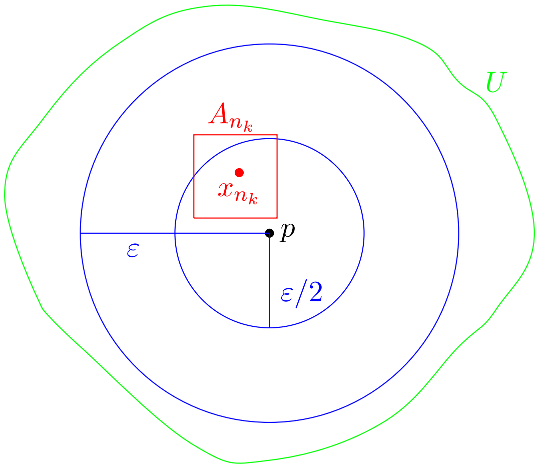
For each $n = 1, 2, \ldots$, choose a point $x_n \in A_n$. Then the sequence $\{x_n\}$ contains

- either finitely many distinct points (in which case some point repeats infinitely times)
- or infinitely many distinct points (in which case $\{x_n\}$ has a limit point since X is compact).

Denote the repeated point, or limit point, by p. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to p. Since $p \in X$ and $\mathscr{F}$ is an open cover of X, there is an open set $U \in \mathscr{F}$ containing p. Choose $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$, and choose an integer k large enough so that:

(a) $d(A_{n_k}) < \varepsilon/2 \implies d(x_{n_k}, x) < \varepsilon/2$ for all $x \in A_{n_k}$, and

(b) $d(x_{n_k}, p) < \varepsilon/2 \iff x_{n_k} \in B_{\varepsilon/2}(p)$.



Thus we have

$$d(x,p) \le d(x,x_{n_k}) + d(x_{n_k},p) < \varepsilon \quad \text{for all } x \in A_{n_k} \implies A_{n_k} \subseteq B_\varepsilon(p) \subseteq U.$$

This contradicts our initial choice of the sequence $\{A_n\}$.

Definition A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be *uniformly continuous* if given $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \varepsilon.$$

Theorem 27.6 (Uniform Continuity Theorem) Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proof Given $\varepsilon > 0$, take the open covering $\{B(y, \varepsilon/2) \mid y \in Y\}$ of Y by balls $B(y, \varepsilon/2)$ of radius $\varepsilon/2$. Let $\mathscr{F} = \{f^{-1}(B(y, \varepsilon/2)) \mid y \in Y\}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathscr{F}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \varepsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \varepsilon.$, as desired.

§23 Connected Spaces

Definition Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

A space X is *disconnected* if there exists a separation U, V of X.

Lemma 23.1 Let A and B be subsets of a topological space X. Then A and B of X form a separation of X, i.e. X is disconnected, if

$$A \neq \emptyset,\ B \neq \emptyset,\ A \cup B = X,\ \bar{A} \cap B = A \cap \bar{B} = \emptyset,$$

Proof If U and V form a separation of X, then $A = U = X \setminus V$ and $B = V = X \setminus U$ are both closed and open subsets of X such that $\bar{A} = A = U \neq \emptyset$, $\bar{B} = B = V \neq \emptyset$, $A \cup B = U \cup V = X$, $\bar{A} \cap B = U \cap V = \emptyset$ and $A \cap \bar{B} = U \cap V = \emptyset$.

Conversely, if $A \neq \emptyset$, $B \neq \emptyset$, $A \cup B = X$, $\bar{A} \cap B = A \cap \bar{B} = \emptyset$, since $\bar{A}, \bar{B} \subseteq X$, we have $\bar{A} \cup B = A \cup \bar{B} = X \implies A = X \setminus \bar{B}$ and $B = X \setminus \bar{A}$ are disjoint nonempty open (and closed) subsets whose union is X, so A and B form a separation of X and X is disconnected.

Remark In fact, a space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Proof ($\Longrightarrow$) If A is a nonempty *proper subset* of X (i.e. $A \subsetneq X$) which is both open and closed in X, then $U = A$ and $V = X \setminus A$ are disjoint nonempty open subsets of X such that $U \cup V = A \cup (X \setminus A) = X$, so U and V form a separation of X and X is disconnected.

($\Longleftarrow$) If U and V form a separation of X, then both $U = X \setminus V$ and $V = X \setminus U$ are nonempty proper open and closed subsets of X.

Remark In summary, the following are equivalent:

(a) X is connected.

(b) X and $\emptyset$ are the only subsets of X which are both open and closed.

(c) X cannot be expressed as the union of two disjoint nonempty open sets.

(d) There are no onto continuous function from X to a discrete space which contains more than one point.

Proof

$[(a) \iff (b)]$ done as in the preceding Remark.

$[(b) \iff (c)]$ done as in the Lemma 23.1.

$[(c) \Rightarrow (d)]$ Suppose (c) is satisfied, and let Y be a discrete space with more than one point and let $f : X \to Y$ be an onto continuous function.

Break up Y as a union $U \cup V$ of two disjoint nonempty open sets. Then $X = \left[f^{-1}(U)\right] \cup \left[f^{-1}(V)\right]$ is the union of two disjoint nonempty open sets, contradicting (c).

$[(d) \Rightarrow (a)]$ Let X be a space which satisfies (d) and suppose X is not connected. There exist $A,\, B \subseteq X$ such that

$$A \neq \emptyset,\ \ B \neq \emptyset,\ \ A \cup B = X \text{ and } \bar{A} \cap B = A \cap \bar{B} = \emptyset.$$

Since $\bar{A}$ $\bar{B}$ are closed in X, $A = X \setminus \bar{B}$ and $B = X \setminus \bar{A}$ are also open in X. Let f be a function from X to the subspace $\{-1,\, 1\}$ of $\mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x \in A \\ \ \ 1 & \text{if } x \in B. \end{cases}$$

Then f is continuous and onto, contradicting (d) for X.

Lemma 23.2 If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

Proof Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y.

These two sets are disjoint and their union is Y; if they were both nonempty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D.

Theorem 23.3 Suppose that $\mathscr{F} = \{A_\alpha\}$ is a collection of connected subspaces of X such that $\bigcap_\alpha A_\alpha \neq \emptyset$. Then the space $Y = \bigcup_\alpha A_\alpha$ is connected.

Proof Let $p \in \bigcap_\alpha A_\alpha$. Suppose that $Y = C \cup D$ is a separation of Y. The point p is in one of the sets C or D; suppose $p \in C$.

Since A_α is connected, it must lie entirely in either C or D, and it cannot lie in D because it contains the point p of C. Hence A_α for every α, so that $A_\alpha \subset C$, contradicting the fact that D is nonempty.

Theorem 23.4 Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$, then B is also connected.

Said differently: If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Proof Let A be connected and let $A \subseteq B \subseteq \bar{A}$. Suppose that $B = C \cup D$ is a separation of B. By Lemma 23.2, the set A must lie entirely in C or in D; suppose that $A \subseteq C$. Then $\bar{A} \subseteq \bar{C}$; since $\bar{C}$ and D are disjoint, $\bar{A}$ and hence $B \subseteq \bar{A}$ cannot intersect D. This contradicts the fact that D is a nonempty subset of B.

Corollary Let Z be a subset of a topological space X. If Z is connected and if Z is *dense* in X (i.e. $\bar{Z} = X$), then X is connected.

Theorem 23.5 Let X be a connected space and let $f : X \to Y$ be a continuous map. Then the image space $Z = f(X)$ is connected.

Proof Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g : X \to Z.$$

Suppose that A is a subset of Z which is both open and closed, then $g^{-1}(A)$ is both open and closed in X. Since X is connected, $g^{-1}(A)$ is either X or $\emptyset$, which implies that A is Z or $\emptyset$. This proves that Z is connected.

Corollary If $h : X \to Y$ is a homeomorphism, then X is connected if and only if Y is connected. In brief, connectedness is a topological property of a space.

Theorem 23.6 If X and Y are connected spaces then the product space $X \times Y$ is connected

Proof Let $a \times b$ be a point in the product space$X \times Y$. Given $x \in X$, since X and Y are connected spaces, X and Y are respectively homomorphic to the "horizontal slice" $X \times \{b\}$ and the "vertical slice" $\{x\} \times Y$, the "T-shaped" space defined by

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y).$$

is connected for each $x \in X$. Furthermore, since $a \times b \in \bigcap_{x \in X} T_x$ (see Figure 23.2), $\bigcup_{x \in X} T_x$ is connected by Theorem 23.3.

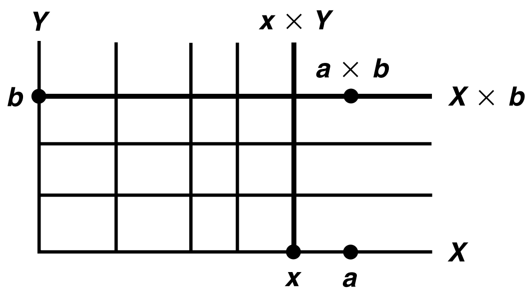


Figure 23.2

Remark By induction and the fact (easily proved) that $X_1 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times \cdots \times X_{n-1}) \times X_n$, one can show that any finite Cartesian product of connected spaces is connected.

§24 Connected Subspaces of the Real Line

Corollary 24.2 The real line $\mathbb{R}$ is a connected space.

Proof Suppose that $\mathbb{R}$ is the union of disjoint nonempty open subsets A and B, that is,

$$A \neq \emptyset,\ B \neq \emptyset,\ A \cup B = \mathbb{R},\ \bar{A} \cap B = A \cap \bar{B} = \emptyset,$$

Choose points $a \in A$, $b \in B$ and suppose for convenience that $a < b$. Let

$$X = \{x \in A \mid x < b\} = A \cap (-\infty, b).$$

Since X is a bounded above nonempty subset of $\mathbb{R} = A \cup B$, the least upper bound $s = \sup X$ exists, and s is in either A or B since $s \in \mathbb{R} = A \cup B$. However,

- if $s \in A$, since $b \in B$ and $s = \sup X$, we must have $s \lneq b$ and $(s, b) \subseteq B$ which implies that $s \in B' \subseteq \bar{B}$ and $s \in A \cap \bar{B} \neq \emptyset$, contrary to the assumption that $A \cap \bar{B} = \emptyset$;
- if $s \in B$, since $s = \sup X$, $s \in X' \subseteq A' \subseteq \bar{A}$ and $s \in \bar{A} \cap B \neq \emptyset$, contrary to the assumption that $\bar{A} \cap B = \emptyset$.

So, $s \notin A \cup B$ which contradicts the fact that $s \in \mathbb{R} = A \cup B$.

Remark If we replace the real line $\mathbb{R}$ by an interval I in the proof, we can show that any interval I is connected.

Remark Let X be a nonempty subset of $\mathbb{R}$. Then X is connected if and only if X is an interval.

Proof ($\Longrightarrow$) Suppose that X is not an interval, there exist $a < b \in X \subset \mathbb{R}$ and $p \notin X$ such that $a < p < b$, $A = \{x \in X \mid x < p\}$ and $B = X \setminus A = \{x \in X \mid p < x\}$ are disjoint nonempty subsets of X whose union is X.

Since

- A is a bounded above nonempty subset of $\mathbb{R}$, the least upper bound $s = \sup A$ exists, $s \in \bar{A}$ and $s \leq p \implies s \notin B \implies \bar{A} \cap B = \emptyset$,
- B is a bounded below nonempty subset of $\mathbb{R}$, the greatest lower bound $m = \inf B$ exists, $b \in \bar{B}$ and $m \geq p \implies m \notin A \implies A \cap \bar{B} = \emptyset$,

so A and B form a separation of X and X is disconnected.

Theorem 24.3 **(Intermediate value theorem)** Let $f : X \to \mathbb{R}$ be a continuous map, where X is a connected space and $\mathbb{R}$ is in the usual (order) topology. If a and b are two points of X and if r is a point of $\mathbb{R}$ lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof Assume the hypotheses of the theorem. The sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, \infty)$$

are disjoint and nonempty because $f(a) \in A$ and $f(b) \in B$. Also, since $(-\infty, r)$, (r, ∞) are open in $\mathbb{R}$, the sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$ are open subsets in $f(X)$. If there were no point c of X such that $f(c) = r$, then $f(X)$ would be the union of the sets A and

B. Then A and B would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected.

Definition Given points x and y of the topological space X, a *path* in X from x to y is a continuous function $\gamma : [a, b] \to X$ of some closed interval in the real line into X, such that $\gamma(a) = x$ and $\gamma(b) = y$. A space X is said to be *path-connected* if every pair of points of X can be joined by a path in X.

Theorem If X is a path-connected space, then X is connected.

Proof Suppose $X = A \cup B$ is a separation of X. Let $\gamma : [a, b] \to X$ be any path in X. Since γ is continuous and $[a, b]$ is connected, the set $\gamma([a, b])$ is connected and it lies entirely in either A or B. Therefore, there is no path in X joining a point of A to a point of B, contrary to the assumption that X is path connected.

Theorem If X is a connected open subset of the Euclidean space $\mathbb{E}^n$, then X is path-connected.

Proof Given $x \in X$, let $U(x)$ be the collection of points of X defined by

$$U(x) = \{y \in X \mid y \text{ can be joined to } x \text{ by a path in } X\}.$$

Then $U(x) \neq \emptyset$ and $U(x)$ is a path connected subset (component) of X.

Claim For each $x \in X$, $U(x)$ is open in X.

Proof of Claim Let $y \in U(x)$, since X is open in $\mathbb{E}^n$, there exists a ball $B_r(y)$ such that $B_r(y) \subseteq X$. If $z \in B_r(y)$, since z can be joined to x by a path in X and $U(x)$ is a path-connected component of X, we must have $z \in U(x)$ and $B_r(y) \subseteq X$. This implies that $U(x)$ is open in X.

Claim For each $x \in X$, $U(x)$ is closed in X.

Proof of Claim Since

$$X \setminus U(x) = \bigcup_{y \in X \setminus U(x)} U(y) = \text{ union of open subset } U(y) \text{ of } X,$$

$X \setminus U(x)$ is open in X and thus $U(x)$ is closed in X.

Since X is connected and $U(x) \neq \emptyset$ is both open and closed in X, we must have $U(x) = X$ which implies that X is path-connected.

The converse is not true: the *topologist's sine curve* is connected but not path-connected.

Example Let $Z = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$, $Y = \{0\} \times [-1, 1]$ and $X = Y \cup Z = \bar{Z} \subset \mathbb{R}^2$ be the topologist's sine curve.

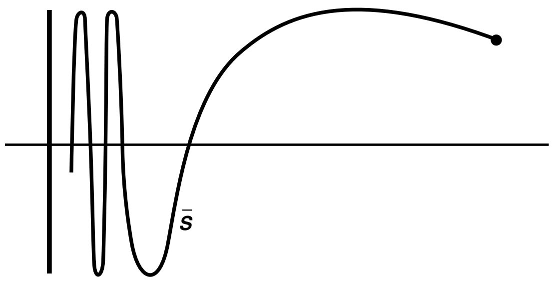


Figure 24.5

Since Z is path-connected, it is connected and $\bar{Z} = X$ is connected.

Suppose that X is path connected and $f : [0,1] \to X$ is a one-to-one continuous function such that $f(0) = (0,0)$ and $f(1) = (1, \sin 1)$, i.e. $f : [0,1] \to X$ is a path beginning at $(0,0) \in Y$ and ending at $(1, \sin 1) \in Z$.

Let π_x, $\pi_y : X \to \mathbb{R}$ be projection maps to the $x-$ and y-coordinates respectively. Since $\pi_x \circ f$ is continuous on $[0,1]$ and by the intermediate value theorem,

$$\pi_x \circ f(0) = 0 < 1 = \pi_x \circ f(1) \implies (\pi_x \circ f)([0,1]) = [0,1] \implies Z \subset f([0,1]).$$

For each $n = 0, 1, 2, \ldots$, since $\left((2n\pi + \pi/2)^{-1}, 1\right) \in Z \subset f([0,1])$ and f is one-to-one, there exists a strictly decreasing sequence $\{t_n\} \subset (0,1]$ such that

$$f(t_n) = (\pi_x \circ f(t_n), \pi_y \circ f(t_n)) = \left((2n\pi + \pi/2)^{-1}, 1\right).$$

Since $[0,1]$ is compact, $\{t_n\}$ is a strictly decreasing sequence in $[0,1]$ and $\pi_y \circ f$ is continuous on $[0,1]$, there exists a $t \in [0,1]$ such that

$$\lim_{n\to\infty} t_n = t \in [0,1], \quad \text{and} \quad \pi_y \circ f(t) = \lim_{n\to\infty} \pi_y \circ f(t_n) = 1.$$

Furthermore, since $\pi_y \circ f$ is continuous on $[0,1]$ and $(\pi_y \circ f)([t_{n+1}, t_n]) = [-1,1]$ for each $n = 0, 1, 2, \ldots$, there exists $u_n \in (t_{n+1}, t_n)$ such that

$$\pi_y \circ f(u_n) = -1, \quad \lim_{n\to\infty} u_n = t, \quad \text{and} \quad \pi_y \circ f(t) = \lim_{n\to\infty} \pi_y \circ f(u_n) = -1$$

which is a contradiction to the continuity of $\pi_y \circ f$. Hence, there does not exist a path joining from a point in Y to Z, i.e. the topologist's since curve $X = \bar{Z}$ is not path connected.

§25 Components and Local Connectednes

Definition An *equivalence relation* $\sim$ on a set X is a relation having the following three properties:

- (Reflexivity) $x \sim x$ for every $x \in X$.
- (Symmetry) If $x \sim y$, then $y \sim x$.
- (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

The *equivalence class* $[x]$ of an element $x \in X$ is the set defined by

$$[x] = \{y \in X \mid y \sim x\}.$$

It is easy to see that distinct equivalence classes are disjoint, i.e $[x] \cap [y]$ is either $\emptyset$ or all of $[x]$.

Definition Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence class C_x of an element $x \in X$ is called a *component* (or "*connected component*") of X.

Remark By the Theorem 23.3, we have the following.

- Let C and D be connected subsets of the space X. If $C \cap D \neq \emptyset$, then $C \cup D$ is connected.
- For each $x \in X$, the (connected) component C_x is the largest connected subset containing of x.

Theorem Let X be a topological space and let C_x denote the component of X containing $x \in X$. Then

- For each $x \in X$, the component C_x is closed in X.
- For any $x,\, y \in X$, $C_x \cap C_y$ is either an empty set or all of $C_x = C_y$. Hence, two components are either disjoint or identical, that is,

$$\forall\, x,\, y \in X, \quad \text{either} \quad C_x \cap C_y = \emptyset, \quad \text{or} \quad C_x = C_y.$$

Proof Let C_x be a component of X containing x. Then C_x is connected, and so $\bar{C}_x$ is connected by a preceding Corollary. Since C_x is an equivalence class of X, we must have $C_x = \bar{C}_x$ and C_x is closed.

If C_x, C_y are components of X such that $C_x \cap C_y \neq \emptyset$ then, since $C_x \cup C_y$ is a connected subset of X containing both C_x and C_y, we must have $C_x \cup C_y = C_x$ and $C_x \cup C_y = C_y$ which implies that $C_x = C_y$. So, distinct components are separated from one another in the space.

Theorem 25.1 The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

Definition We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the *path components* (or "*path-connected components*") of X.

This is an equivalence relation since

- for each $x \in X$, the path γ defined by

$$\gamma(t) = x \quad t \in [a, b]$$

  is a path in X joining x to x, i.e. the relation $\sim$ is reflexive;
- if γ is a path in X joining x to y, then $-\gamma$ defined by

$$-\gamma(t) = \gamma(a + b - t) \quad t \in [a, b]$$

  is a (reversed) path in X joining y to x, i.e. the relation $\sim$ is symmetric;
- if α, β are paths in X joining x to y and y to z respectively, then γ defined by

$$\gamma(t) = \begin{cases} \alpha(2t - a) & \text{if} \quad a \leq t \leq (a + b)/2, \\ \beta(2t - b) & \text{if} \quad (a + b)/2 \leq t \leq b. \end{cases}$$

  is a path in X joining x to z, i.e. the relation $\sim$ is transitive.

Theorem 25.2 The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.